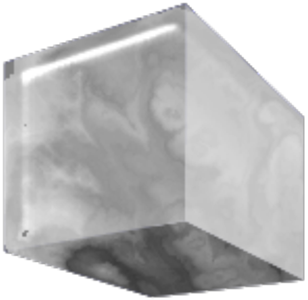
2018.07.07 LFMTP

Cubical Computational Type Theory & RedPRL



>> redprl.org >>

Carlo Angiuli
Evan Cavallo
(*) Favonia
Robert Harper
Jonathan Sterling
Todd Wilson

Cubical

features of homotopy type theory

univalence, higher inductive types

+

Computational

features of Nuprl and PVS

strict equality, strict quotients,
predicative subtypes...

Cartesian Cubical

features of homotopy type theory

univalence, higher inductive types

+

Computational

features of Nuprl and PVS

strict equality, strict quotients,
predicative subtypes...

Computational Types

programs/
realizers

computation

Computational Types

programs/ realizers	<------	computational type theory
computation		theory of computation

Computational Types

programs/ realizers <----- **computational type theory**

computation — theory of computation

---

meaning explanation <---- Martin-Löf type theory

pre-mathematical in M-L's work

A Minimum Example

```
M := a | bool | true | false | if(M,M,M)
```

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bool val       if(M,Mt,Mf) ↦ if(M',Mt,Mf)
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false val      if(false,_,M) ↦ M
```

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The Language

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What are the types in canonical forms? **{bool}**

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What are the types in canonical forms? **{bool}**

What are the canonical forms of the types?

bool: {true, false}

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What are the types in canonical forms? **{bool}**

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How they are equal? **syntactic equality**

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What are the types in canonical forms? **{bool}**

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One Theory

A Minimum Example

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M := a | bool | true | false | if(M,M,M)
```

types: {bool} with syntactic equality $\approx$

bool: {true, false} with syntactic equality $\approx_{\text{bool}}$

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$$A \doteq B \text{ type}$$

$A \Downarrow A'$ $B \Downarrow B'$ and $A' \approx B'$

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$\text{bool} \doteq \text{bool type}$

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---

bool $\doteq$ bool type

if(true,bool,bool) $\doteq$ bool type

$\Downarrow$ bool

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if(true,bool,bool) $\doteq$ bool type

$\Downarrow$bool

if(true,bool,*any closed term*) $\doteq$ bool type

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$$M \doteq N \in A$$

$A \doteq A$ type, $M \Downarrow M'$, $N \Downarrow N'$, $A \Downarrow A'$ and $M' \approx_{A'} N'$

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$$\text{false} \doteq \text{false} \in \text{bool}$$

A Minimum Example

$M := a \mid \mathrm{bool} \mid \mathrm{true} \mid \mathrm{false} \mid \mathrm{if}(M,M,M)$

types: {bool} with syntactic equality $\approx$

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$$M \doteq N \in A$$

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$$\mathrm{false} \doteq \mathrm{false} \in \mathrm{bool}$$

$$\underbrace{\mathrm{if}(\mathrm{true},\mathrm{true},\mathrm{bool})}_{\Downarrow \mathrm{true}} \doteq \mathrm{true} \in \underbrace{\mathrm{if}(\mathrm{true},\mathrm{bool},\mathrm{bool})}_{\Downarrow \mathrm{bool}}$$

A Minimum Example

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$$a{:}A \gg M \doteq N \in B$$

$P \doteq Q \in A$ implies $M[P/a] \doteq N[Q/a] \in B$

A Minimum Example

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$$a{:}A \gg M \doteq N \in B$$

$$P \doteq Q \in A \text{ implies } M[P/a] \doteq N[Q/a] \in B$$

$$b{:}\text{bool} \gg b \doteq \text{if}(b,\text{true},\text{false}) \in \text{bool}?$$

A Functional Example

```
M := a | M1⟶M2 | \a.M | M1 M2 | ...

(M1⟶M2) val  \a.M val  (\a.M1)M2 ↦ M1[M2/a]
```

Another Language

A Functional Example

```
M := a | M1→M2 | \a.M | M1 M2 | ...

(M1→M2) val  \a.M val  (\a.M1)M2 ↦ M1[M2/a]
```

Another Language

What are the types in canonical forms?

```
the least fixed point of
S ↦ {M→N | M⇓, N⇓ in S} union ...
```

What are the canonical forms of the types?

```
A→B: {\a.M}
```

How they are equal?

```
A1→B1 ≈ A2→B2 if A1 ≐ A2 and B1 ≐ B2
```

$\backslash a.M1 \approx_{A\to B} \backslash a.M2$ if $a{:}A \gg M1 \doteq M2 \in B$

Variables

Nuprl/...	Coq/Agda/...
Vars range over closed terms	Vars are indet.
Defined by transition b/w closed terms	Defined by conversion b/w open terms

Open-endedness

Proof theory/tactics/editors

$\downarrow$

Computational type theory

$\downarrow$

Programming language

Open-endedness

Proof theory/tactics/editors

$\downarrow$

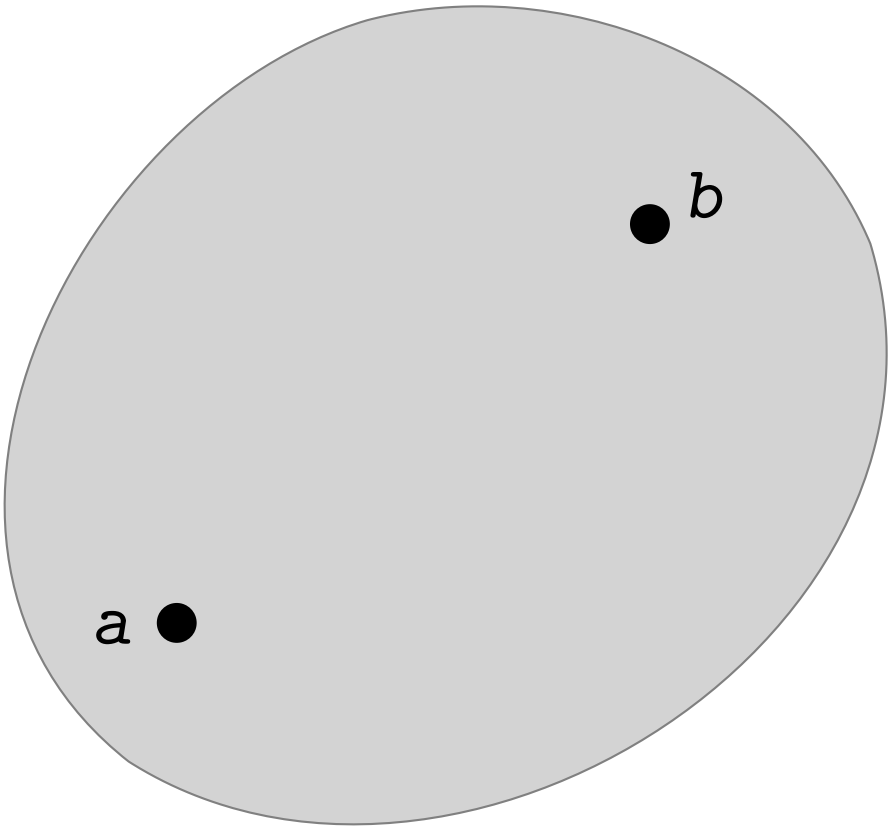
Computational type theory

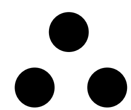
$\downarrow$

Programming language

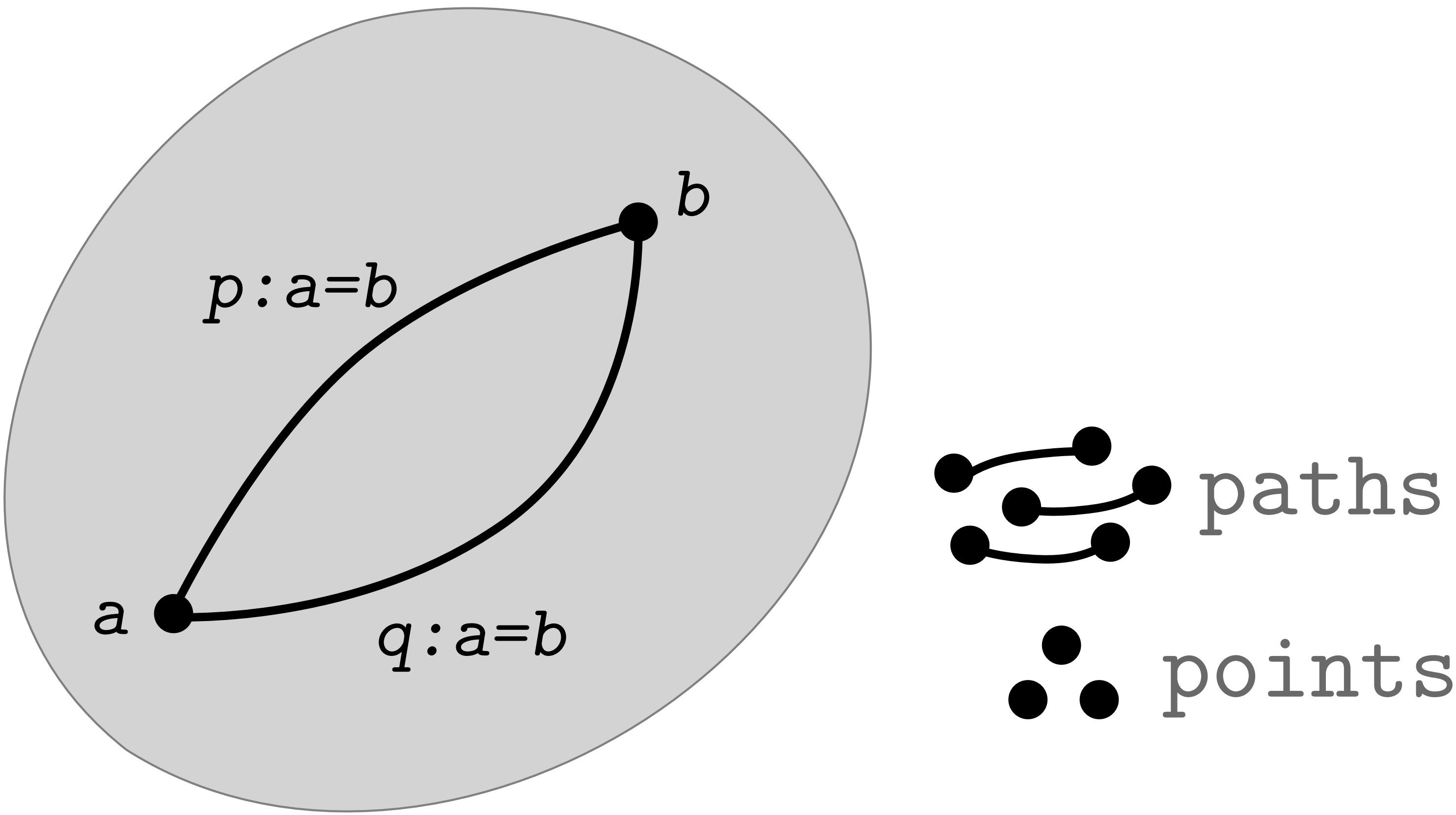
Canonicity always holds

Homotopy Type Theory

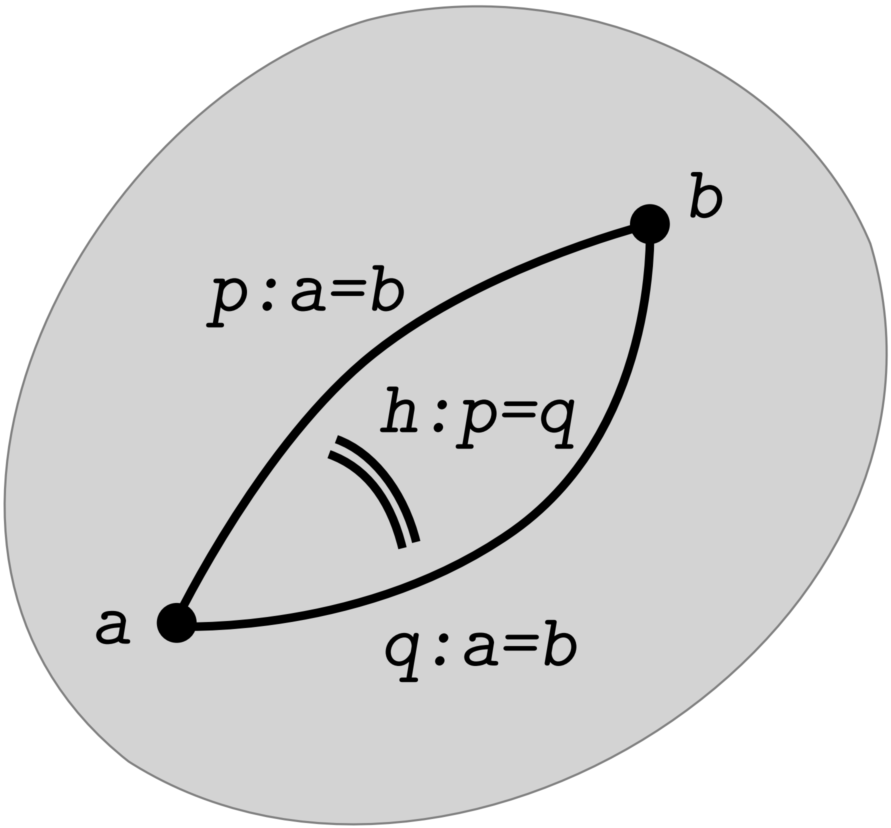


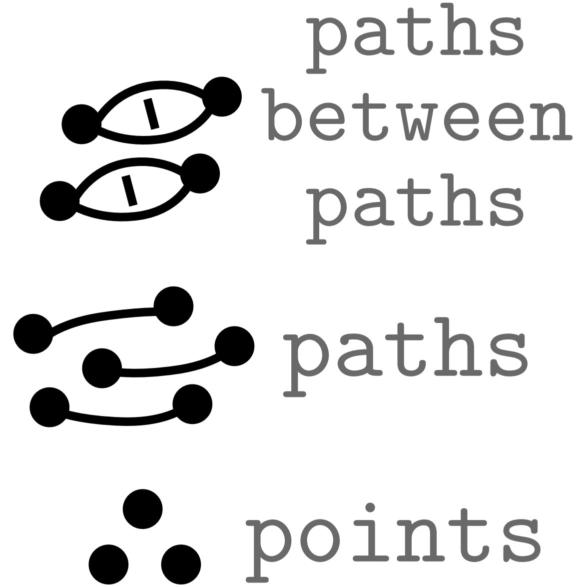


Homotopy Type Theory

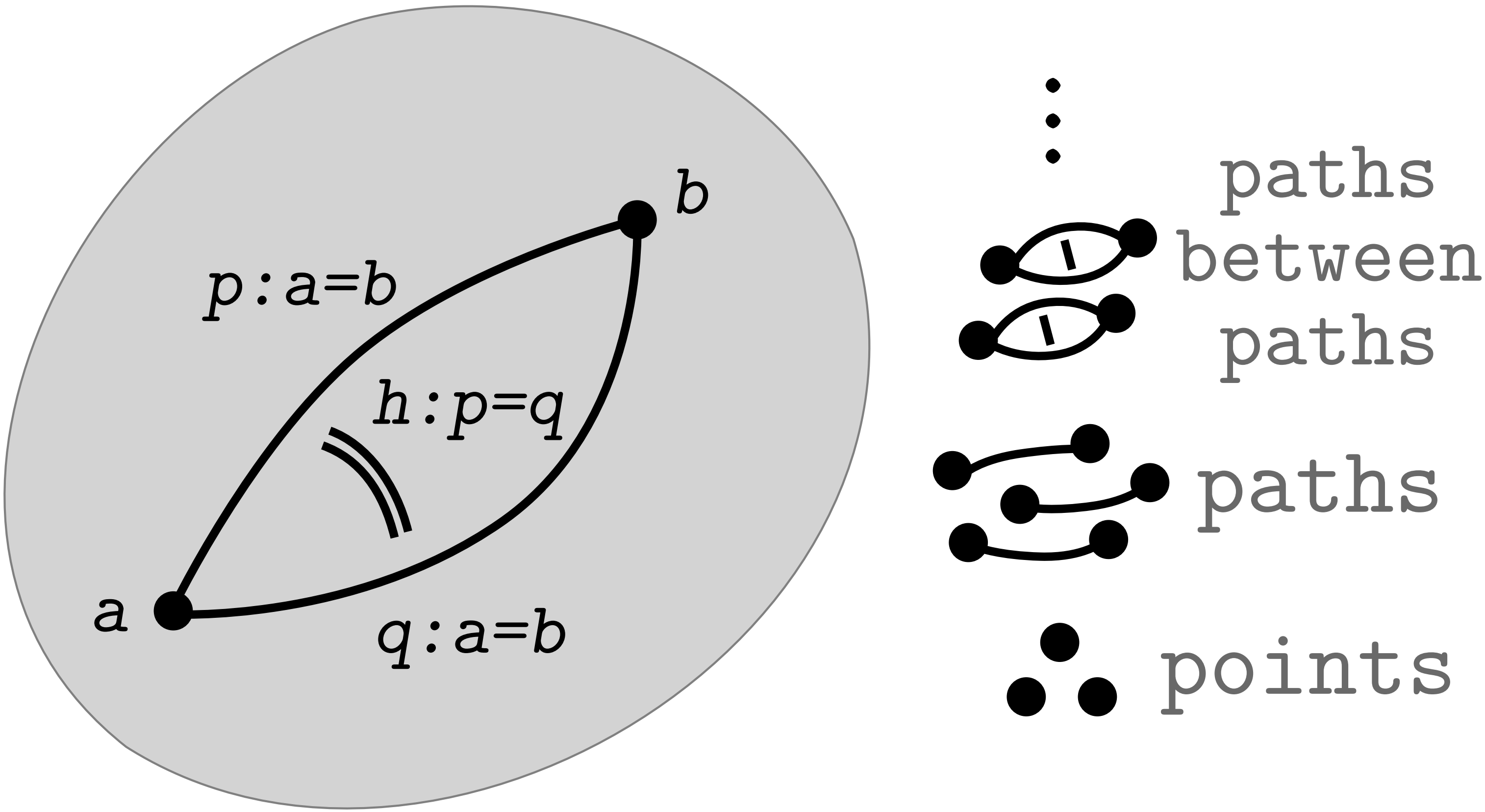


Homotopy Type Theory





Homotopy Type Theory



Equality and Paths

Equality ($\equiv$)

Silent in theory

$$2 + 3 \equiv 5$$
$$fst\ \langle M, N \rangle \equiv M$$

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If $A \equiv B$ and $M : A$ then $M : B$

Equality and Paths

Equality ($\equiv$)

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$2 + 3 \equiv 5$

$fst\ \langle M,N \rangle \equiv M$

If $A \equiv B$ and $M : A$ then $M : B$

Paths ($=$)

Visible in theory

If $P : A = B$ and $M : A$ then $transport(M,P) : B$

Homotopy Type Theory

[Awodey and Warren] [Voevodsky *et al*] [van den Berg and Garner]

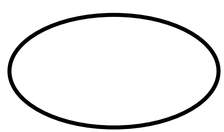
A	Type	Space
$a : A$	Element	Point
$f : A \to B$	Function	Continuous Mapping
$C : A \to Type$	Dependent Type	Fibration
$a =_A b$	Identification	Path

Features of HoTT

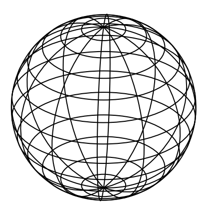
Univalence

If e is an equivalence between types A and B, then $ua(E):A=B$

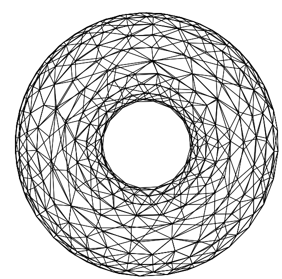
Higher Inductive Types



circle



sphere



torus

Canonicity?

Canonicity broken by
new features stated as axioms!

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Canonicity

For any $M : bool$, either
$M \equiv true : bool$ or $M \equiv false : bool$

Canonicity?

Canonicity broken by
new features stated as axioms!

Canonicity

For any $M : bool$, either
$M \equiv true : bool$ or $M \equiv false : bool$

$ua(not) : bool = bool$

$transport(ua(not), true) \not\equiv false$

Canonicity for All

Canonicity for bool means
canonicity for *everyone*

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$$M : bool \times A$$
$$fst(M) \equiv ??? : bool$$

Canonicity for All

Canonicity for bool means
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Wants $M \equiv \langle P, Q \rangle$ and then
$fst(M) \equiv fst\langle P, Q \rangle \equiv P \equiv$ true or false

Canonicity for Paths?

$$\frac{M : A}{\mathit{refl}(M) : M =_A M}$$

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$$\frac{M : A}{\mathit{refl}(M) : M =_A M}$$

$$\frac{a{:}A \vdash R : C(a,a,\mathit{refl}(a)) \qquad P : M = N}{\mathit{path\text{-}ind}[C](a.R,P) : C(M,N,P)}$$

Canonicity for Paths?

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$$\frac{a{:}A \vdash R : C(a,a,\mathtt{refl}(a)) \qquad M : A}{\mathtt{path\text{-}ind}[C](a.R,\mathtt{refl}(M)) \equiv R[M/a] \\ : C(M,M,\mathtt{refl}(M))}$$

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$$\frac{a{:}A \vdash R : C(a,a,\mathit{refl}(a)) \quad M : A}{\mathit{path\text{-}ind}[C](a.R,\mathit{refl}(M)) \equiv R[M/a] : C(M,M,\mathit{refl}(M))}$$

$$\mathit{path\text{-}ind}[C](a.R,\mathit{ua}(E)) \equiv \; ???$$

Restore Canonicity

Can we have a new TT with canonicity + univalence?

Yes with De Morgan cubes [CCHM 2016]
Yes with Cartesian cubes [AFH 2017]

... and higher inductive types?

Examples with De Morgan cubes [CHM 2018]
Yes with Cartesian cubes [CH 2018]

Restore Canonicity

Idea: each type manages its own paths

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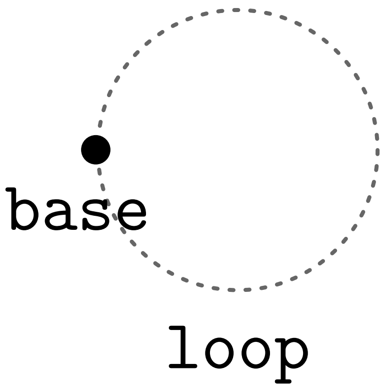
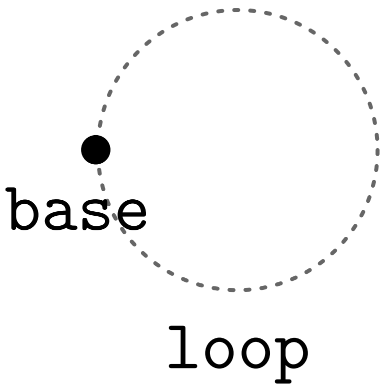


base : S1

Restore Canonicity

Idea: each type manages its own paths




```
base : S1
loop : base = base
```

Restore Canonicity

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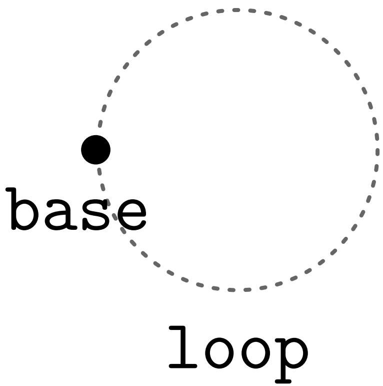

`base : S1`

~~`loop : base = base`~~

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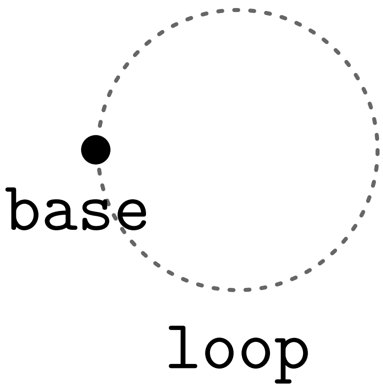


base : S1

~~loop : base = base~~

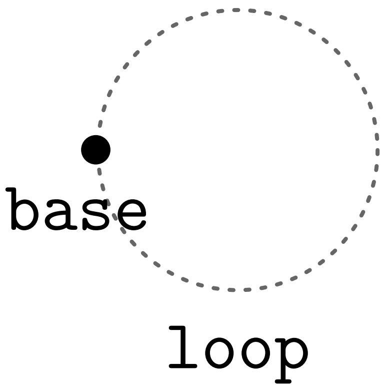
$x{:}\mathbb{I} \vdash$ loop{x} : S1

loop{0} $\equiv$ base : S1

loop{1} $\equiv$ base : S1

Restore Canonicity

Idea: each type manages its own paths



base : S1

~~loop : base = base~~

x:$\mathbb{I}$ ⊢ loop{x} : S1

loop{0} ≡ base : S1

loop{1} ≡ base : S1

Kan structure:
sufficient to implement path-ind

Kan types: types with Kan structure

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

$$\Gamma \vdash 0 : \mathbb{I} \qquad \Gamma \vdash 1 : \mathbb{I}$$

$$\Gamma,\ x : \mathbb{I},\ \Gamma' \vdash x : \mathbb{I}$$

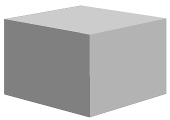
Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

$$\Gamma \vdash 0 : \mathbb{I} \qquad \Gamma \vdash 1 : \mathbb{I}$$

$$\Gamma,\ x:\mathbb{I},\ \Gamma' \vdash x:\mathbb{I}$$

$$x_1:\mathbb{I},\ x_2:\mathbb{I},\ \ldots,\ x_n:\mathbb{I} \vdash M : A$$

$\Leftrightarrow$ M is an n-cube in A 

Cartesian Cubes

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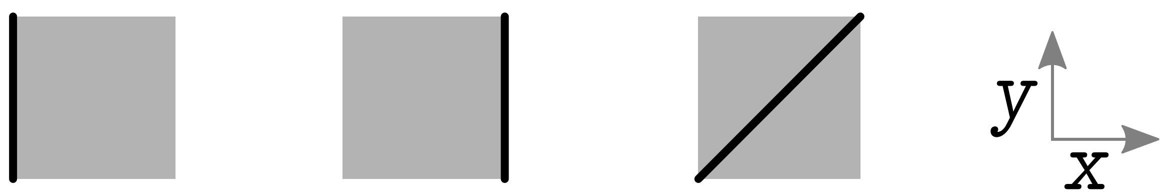
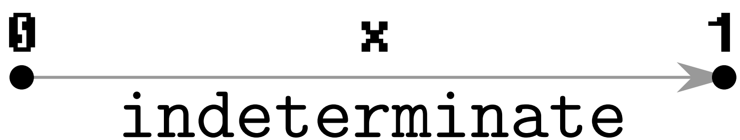
Cartesian: works as normal contexts

$$M\langle 0/x\rangle \qquad M\langle 1/x\rangle \qquad M\langle y/x\rangle$$

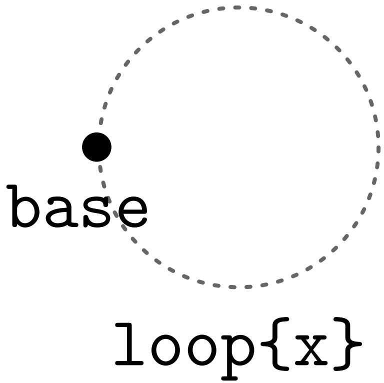


Cubical Programming

```
dim expr r := 0 | 1 | x
```



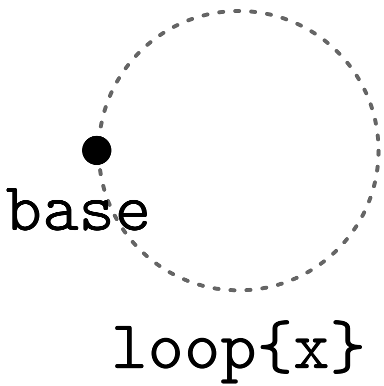
Circle



Circle

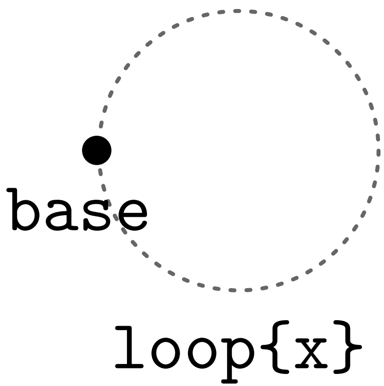
```
M := S1 | base | loop{r}
   | S1elim(a.M, M, M, x.M) | ...
```

dim
expr



Circle

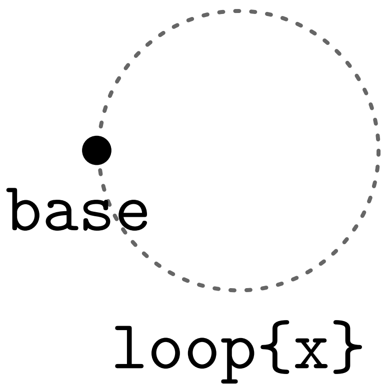
```
                                       dim
M := S1 | base | loop{r}              expr
   | S1elim(a.M, M, M, x.M) | ...
```



S1 val

Circle

```
M := S1 | base | loop{r}    dim
                            expr
   | S1elim(a.M, M, M, x.M) | ...
```



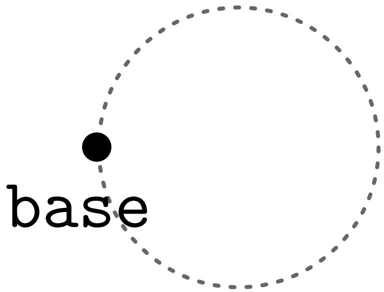
`S1 val`

`base val`

Circle

```
M := S1 | base | loop{r}
   | S1elim(a.M, M, M, x.M) | ...
```

dim expr



`S1 val`

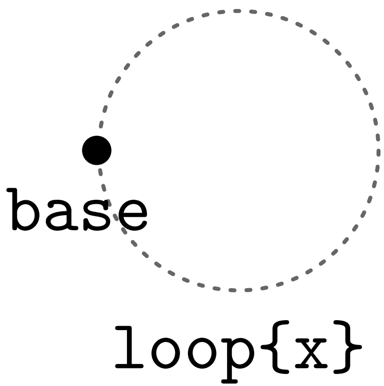
`base val`

`loop{x} val`

`loop{0} ↦ base`

`loop{1} ↦ base`

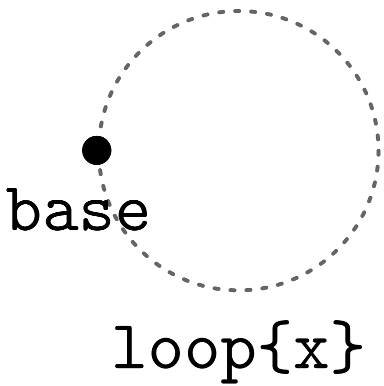
Circle




```
M ↦ M'
────────────────────────────
S1elim(a.A, M, B, x.L)
  ↦ S1elim(a.A, M', B, x.L)
```

```
S1 val
```

Circle

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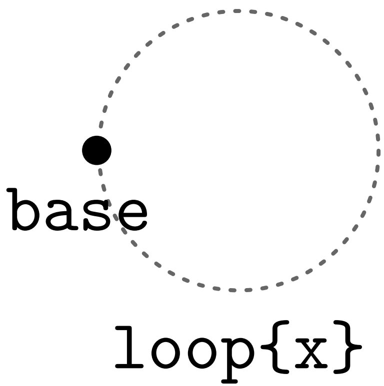
  ↦ S1elim(a.A, M', B, x.L)
```

```
S1elim(a.A, base, B, x._)
  ↦ B
```

Circle

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```
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———————————————————————
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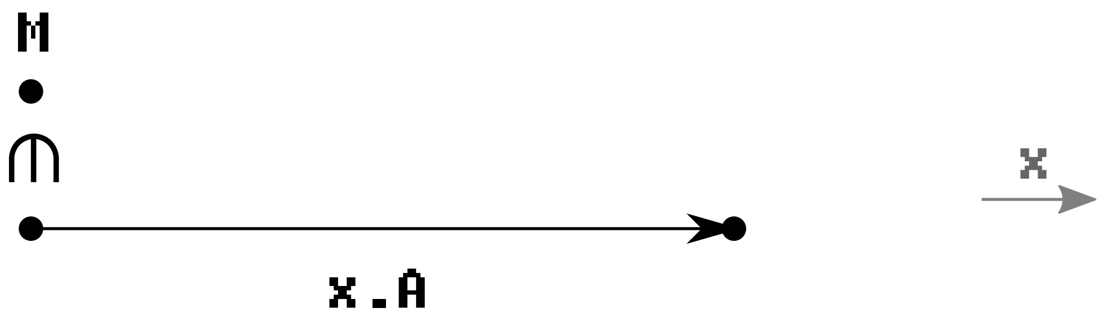
  ↦ S1elim(a.A, M', B, x.L)
```

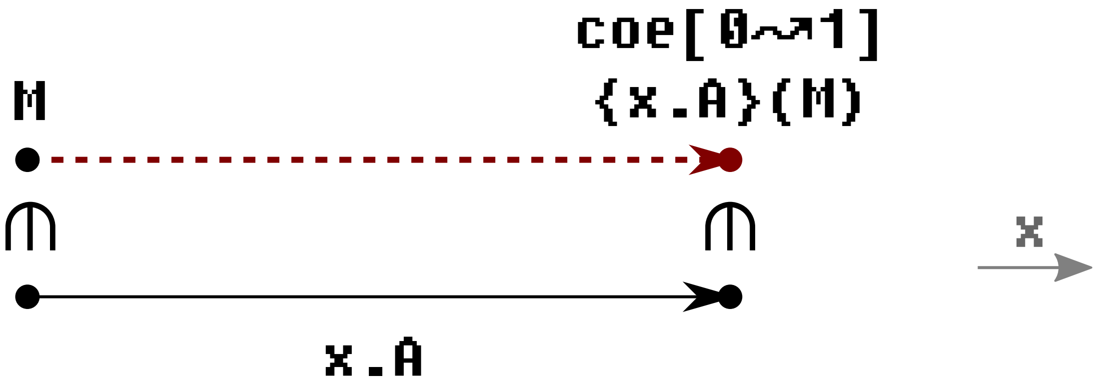
```
S1elim(a.A, base, B, x._)
  ↦ B
```

```
S1elim(a.A, loop{x}, _, y.L)
  ↦ L<x/y>
```

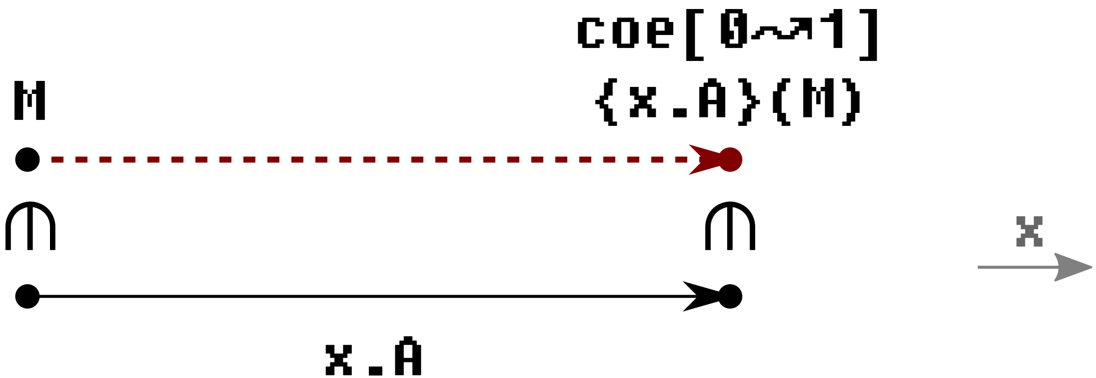
Kan 1/2: Coercion



Kan 1/2: Coercion



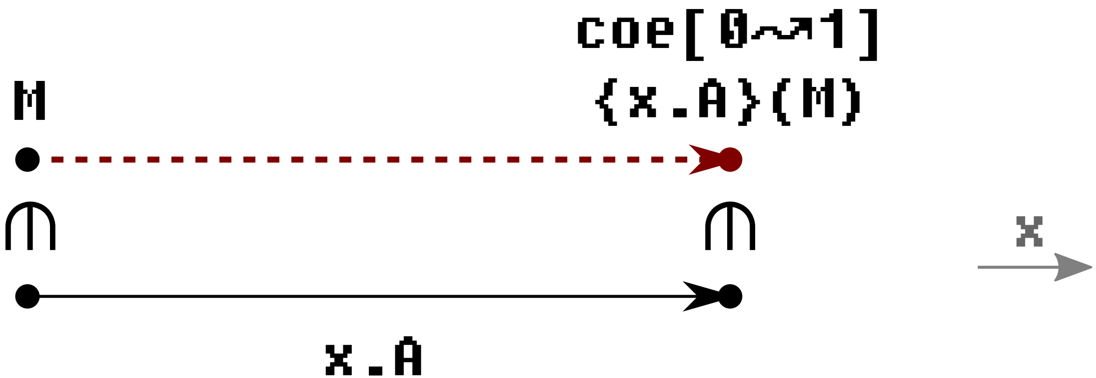
Kan 1/2: Coercion



coe[r⤳r']{x.A}(M) ∈ A<r'/x>

A<r/x>

Kan 1/2: Coercion



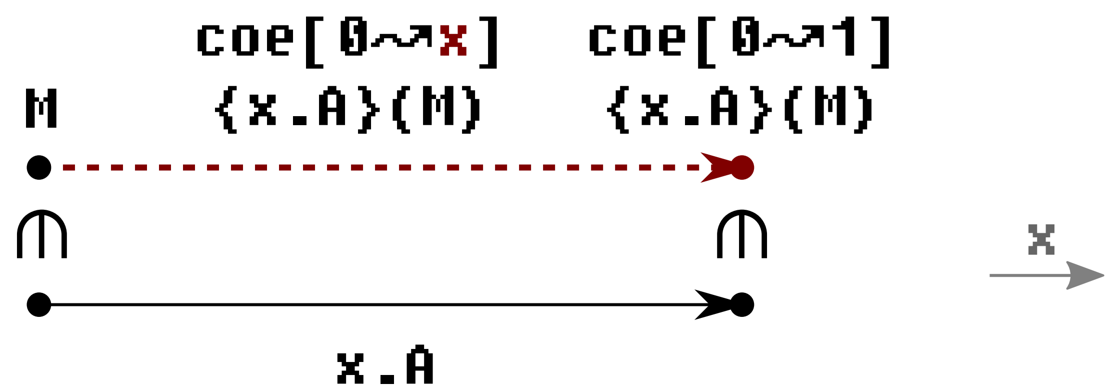
$$\mathtt{coe}[r \rightsquigarrow r']\{x.A\}(M) \in A\langle r'/x\rangle$$

$$\Cap$$

$$A\langle r/x\rangle$$

$$\mathtt{coe}[r \rightsquigarrow r]\{x.A\}(M) \doteq M \in A\langle r/x\rangle$$

Kan 1/2: Coercion



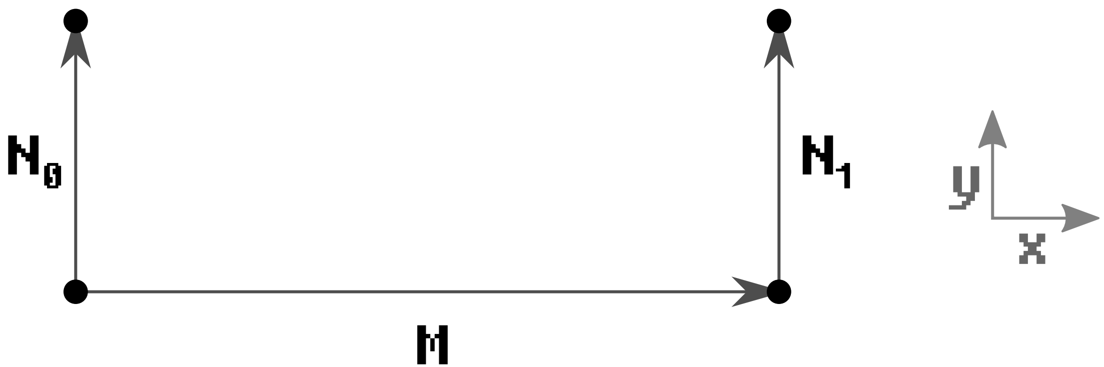
coe[r⇝r']{x.A}(M) ∈ A<r'/x>

⋒

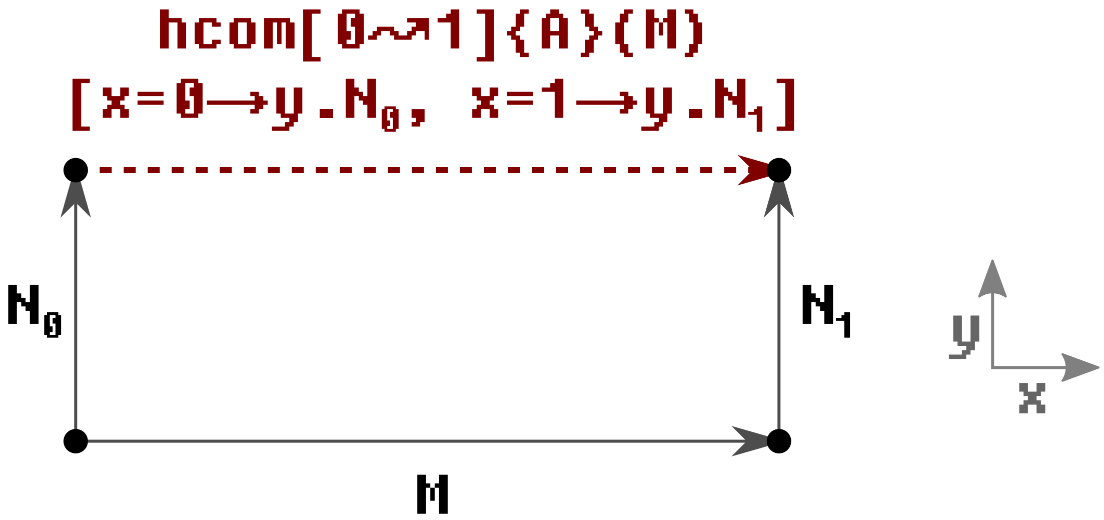
A<r/x>

coe[r⇝r]{x.A}(M) ≐ M ∈ A<r/x>

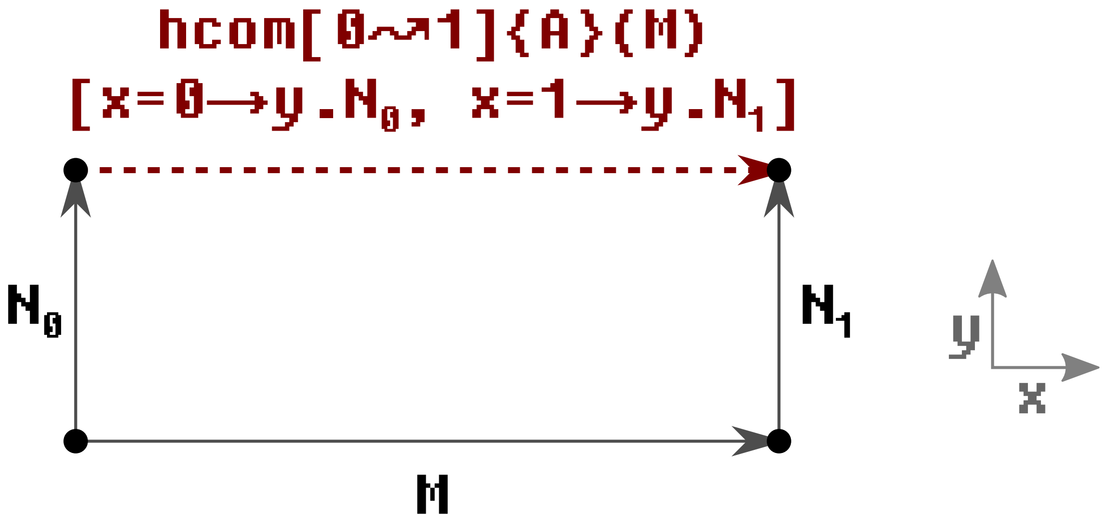
Kan 2/2: Homogeneous Comp.



Kan 2/2: Homogeneous Comp.

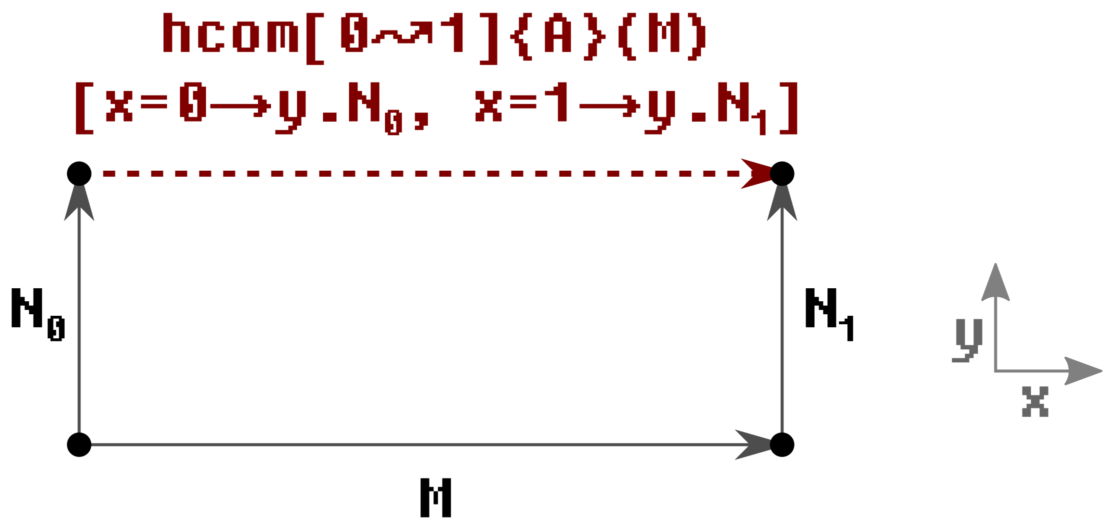


Kan 2/2: Homogeneous Comp.



hcom[r⤳r']{A}(M) […, r_i=r'_i⟶y.N_i, …] ∈ A

Kan 2/2: Homogeneous Comp.

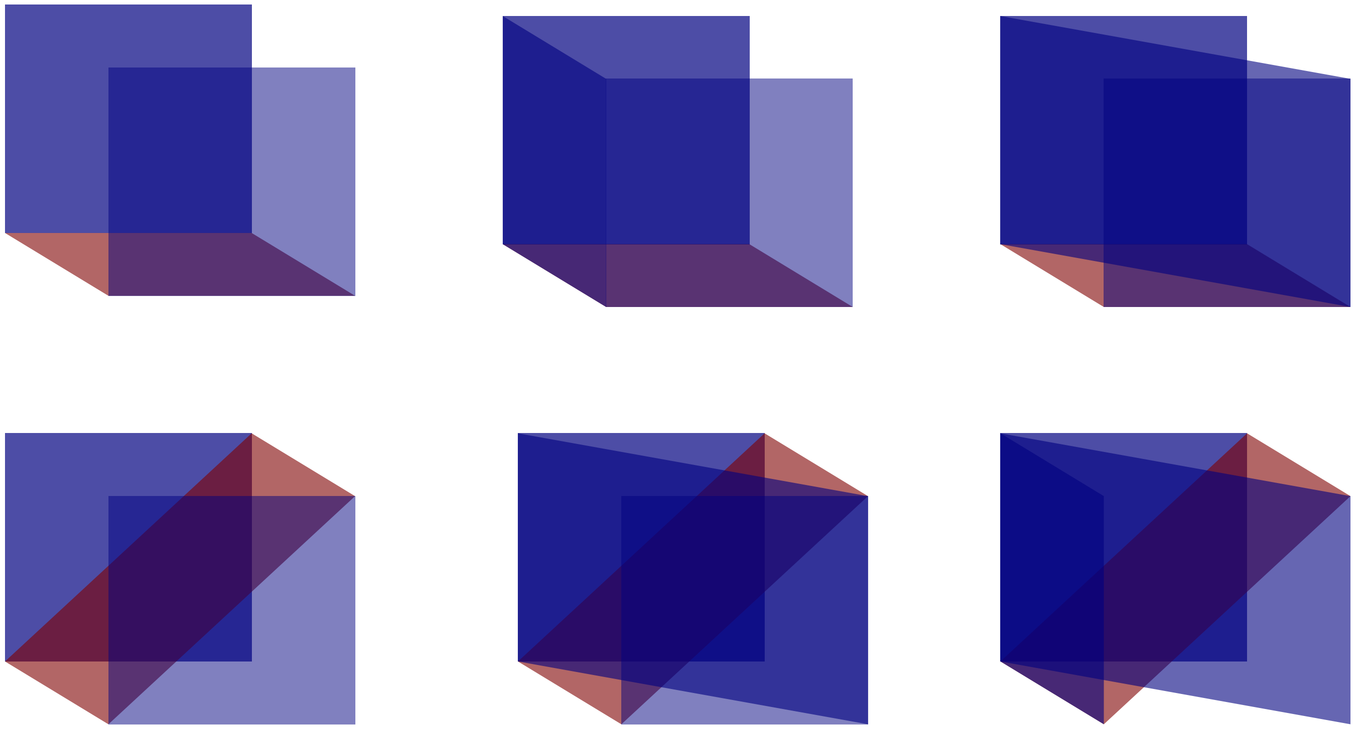


hcom[r⤳r']{A}(M) [..., r$_i$=r'$_i$⟶y.N$_i$, ...] ∈ A

hcom[r⤳r]{A}(M) ≐ M ∈ A

hcom[r⤳r']{A}(M)[..., r$_i$=r$_i$⟶y.N$_i$, ...]
 ≐ N$_i$⟨r'/y⟩ ∈ A

Kan 2/2: Homogeneous Comp.



Kan Circle

```
coe[r~>r'](_.S1)(M) ↦ M
```

Kan Circle

```
coe[r~>r'](_.S1)(M) ↦ M
```

```
hcom[r~>r'](S1)(M)[…] ↦ fhcom[r~>r'](M)[…]
```

formal homo.
composition

Kan Circle

```
coe[r~>r'](_.S1)(M) ↦ M
```

```
hcom[r~>r'](S1)(M)[…] ↦ fhcom[r~>r'](M)[…]
```

formal homo.
composition

```
fhcom[r~>r](M)[…] ↦ M
```

Kan Circle

```
coe[r⇝r'](_.S1)(M) ↦ M
```

```
hcom[r⇝r'](S1)(M)[…] ↦ fhcom[r⇝r'](M)[…]
```

formal homo.
composition

```
fhcom[r⇝r](M)[…] ↦ M
```

```
r!=r'   r_i=r'_i (the first i)
------------------------------------------------
fhcom[r⇝r'](M)[…, r_i=r'_i→y.N_i, …] ↦ N_i<r'/y>
```

Kan Circle

```
coe[r⇝r'](_.S1)(M) ↦ M
```

```
hcom[r⇝r'](S1)(M)[…] ↦ fhcom[r⇝r'](M)[…]
```

formal homo.
composition

```
fhcom[r⇝r](M)[…] ↦ M
```

```
r!=r'   r_i=r'_i (the first i)
------------------------------------------------------
fhcom[r⇝r'](M)[…, r_i=r'_i→y.N_i, …] ↦ N_i<r'/y>
```

```
r!=r'   r_i!=r'_i for all i
----------------------------------
fhcom[r⇝r'](M)[…] val
```

Kan Circle

S1elim needs to handle **fcom**

Kan Circle

`S1elim` needs to handle `fcom`

```
r≠r'   r_i≠r'_i
————————————————————————————————————
S1elim(a.A, fhcom[r⇝r'](M)[…], B, x.L)
  ↦ com[r⇝r']{y.A[fhcom[r⇝y](M)[…]/a}
         (S1elim(M, B, x.L))[…]
```

S1elim(composition) ↦ composition(S1elim)

Cubical Stability

Dimension substs. do not commute with evaluation!

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Dimension substs. do not commute with evaluation!

$$\begin{tikzcd} \mathtt{S1elim(a.A,\ loop\{x\},\ B,\ y.L)} \arrow[r, mapsto, dashed] & \mathtt{L\langle x/y\rangle} \arrow[d, dashed, "\mathtt{\langle 0/x\rangle}"] \\ & \mathtt{L\langle 0/y\rangle} \end{tikzcd}$$

Cubical Stability

Dimension substs. do not commute with evaluation!

$$\begin{tikzcd}
\mathtt{S1elim(a.A,\ loop\{x\},\ B,\ y.L)} \arrow[rr, mapsto, dashed] \arrow[d, "\langle 0/x \rangle"] & & \mathtt{L\langle x/y \rangle} \arrow[d, dashed, "\langle 0/x \rangle"] \\
\mathtt{S1elim(a.A,\ base,\ B,\ y.L)} \arrow[r, mapsto] & \mathtt{B} \arrow[r, phantom, "\langle =??= \rangle"] & \mathtt{L\langle 0/y \rangle}
\end{tikzcd}$$

Cubical Stability

Dimension substs. do not commute with evaluation!

```
S1elim(a.A,
 loop{x}, B, y.L)   |————————→   L<x/y>

       |                            |
       ↓ <0/x>                      ↓ <0/x>

S1elim(a.A,
 base, B, y.L)   |—>  B   <=??=>  L<0/y>
```

Restrict our theory to
only cubically stable parts

Cubical Type Theory

stability: consider every substitution

Cubical Type Theory

stability: consider every substitution

$$A \doteq B \ \mathtt{type}\ [\Psi]$$

dim
context

A and B stably recognize the same stable values
and have stably equal Kan structures

(see our arXiv papers)

Cubical Type Theory

stability: consider every substitution

$$A \doteq B \text{ type } [\Psi]$$

dim context

A and B **stably** recognize the same **stable** values and have **stably equal Kan structures**

$$M \doteq N \in A \ [\Psi]$$

$A \doteq A$ type $[\Psi]$,
M and N **stably** eval to M' and N',
A **stably** treats M' and N' as the same

(see our arXiv papers)

Variables

Nupri/...	Coq/Agda/...
Vars range over closed terms Defined by transition b/w closed terms	Vars are indet. Defined by conversion b/w open terms

exp vars — dim vars

cubical computational TT

arXiv papers

CHTT Part I [AHW 2016]

Cartesian cubical + computational

CHTT Part II [AH 2017]

Dependent types

CHTT Part III [AFH 2017]

Univalent Kan universes

Strict equality

CHTT Part IV [AFH 2017]

Higher inductive types

Proof Assistants

RedPRL

In Nuprl style

redprl.org

redtt

(Work in progress)

github.com/RedPRL/redtt

---

yacctt

Proof of concept

modified from cubicaltt

github.com/mortberg/yacctt

Conclusion

We extended Nuprl semantics
by cubical structure which
justifies key features of HoTT

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We extended Nuprl semantics by cubical structure which justifies key features of HoTT

Best of the two worlds!

We also built proof assistants

redprl.org
github.com/RedPRL/redtt
github.com/mortberg/yacctt