Uniform Atomic Ordered Linear Logic

A Meta-Circular Interpreter for Olli

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Outline

Purely Ordered Logic (Lambek Calculus)

$$\Omega \vdash A$$

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$$\Omega \vdash A$$

$$\frac{}{A \vdash A}\, init$$

Purely Ordered Logic (Lambek Calculus)

$$\Omega \vdash A$$

$$\frac{}{A \vdash A}\, init$$

$$\frac{\Omega, A \vdash B}{\Omega \vdash A \twoheadrightarrow B} \twoheadrightarrow_R \qquad \frac{\Omega_L, B, \Omega_R \vdash C \qquad \Omega_A \vdash A}{\Omega_L, A \twoheadrightarrow B, \Omega_A, \Omega_R \vdash C} \twoheadrightarrow_L$$

Purely Ordered Logic (Lambek Calculus)

$$\Omega \vdash A$$

$$\frac{}{A \vdash A}\,init$$

$$\frac{\Omega, A \vdash B}{\Omega \vdash A \twoheadrightarrow B}\twoheadrightarrow R \qquad \frac{\Omega_L, B, \Omega_R \vdash C \qquad \Omega_A \vdash A}{\Omega_L, A \twoheadrightarrow B, \Omega_A, \Omega_R \vdash C}\twoheadrightarrow L$$

$$\frac{A, \Omega \vdash B}{\Omega \vdash A \rightarrowtail B}\rightarrowtail R \qquad \frac{\Omega_L, B, \Omega_R \vdash C \qquad \Omega_A \vdash A}{\Omega_L, \Omega_A, A \rightarrowtail B, \Omega_R \vdash C}\rightarrowtail L$$

Adding Linear Hypotheses

$$\Delta \,;\, \Omega \;\vdash\; A$$

Adding Linear Hypotheses

$$\Delta \,;\, \Omega \vdash A$$

$$\frac{}{\cdot \,;\, A \vdash A}\,\mathit{init} \qquad \frac{\Delta \,;\, \Omega_L, \boxed{A}, \Omega_R \vdash C}{\Delta \bowtie \boxed{A} \,;\, \Omega_L, \Omega_R \vdash C}\,\mathit{place}$$

$\bowtie$ is non-deterministic merge

Adding Linear Hypotheses

$$\Delta\,;\,\Omega \vdash A$$

$$\frac{}{\cdot\,;\,A \vdash A}\,\mathit{init} \qquad \frac{\Delta\,;\,\Omega_L, A, \Omega_R \vdash C}{\Delta \bowtie A\,;\,\Omega_L, \Omega_R \vdash C}\,\mathit{place}$$

$$\frac{\Delta, A\,;\,\Omega \vdash B}{\Delta\,;\,\Omega \vdash A \multimap B}\,{\multimap}R \qquad \frac{\Delta\,;\,\Omega_L, B, \Omega_R \vdash C \qquad \Delta_A\,;\,\cdot \vdash A}{\Delta \bowtie \Delta_A\,;\,\Omega_L, A \multimap B, \Omega_R \vdash C}\,{\multimap}L$$

Adding Linear Hypotheses

$$\Delta\,;\,\Omega \;\vdash\; A$$

$$\frac{}{\cdot\,;\,A \;\vdash\; A}\,init \qquad \frac{\Delta\,;\,\Omega_L, A, \Omega_R \;\vdash\; C}{\Delta \bowtie A\,;\,\Omega_L, \Omega_R \;\vdash\; C}\,place$$

$$\frac{\Delta, A\,;\,\Omega \;\vdash\; B}{\Delta\,;\,\Omega \;\vdash\; A \multimap B}\,{\multimap}R \qquad \frac{\Delta\,;\,\Omega_L, B, \Omega_R \;\vdash\; C \qquad \Delta_A\,;\,\cdot \;\vdash\; A}{\Delta \bowtie \Delta_A\,;\,\Omega_L, A \multimap B, \Omega_R \;\vdash\; C}\,{\multimap}L$$

$$\frac{\Delta\,;\,\Omega, A \;\vdash\; B}{\Delta\,;\,\Omega \;\vdash\; A \twoheadrightarrow B}\,{\twoheadrightarrow}R \qquad \frac{\Delta\,;\,\Omega_L, B, \Omega_R \;\vdash\; C \qquad \Delta_A\,;\,\Omega_A \;\vdash\; A}{\Delta \bowtie \Delta_A\,;\,\Omega_L, A \twoheadrightarrow B, \Omega_A, \Omega_R \;\vdash\; C}\,{\twoheadrightarrow}L$$

$$\frac{\Delta\,;\,A, \Omega \;\vdash\; B}{\Delta\,;\,\Omega \;\vdash\; A \rightarrowtail B}\,{\rightarrowtail}R \qquad \frac{\Delta\,;\,\Omega_L, B, \Omega_R \;\vdash\; C \qquad \Delta_A\,;\,\Omega_A \;\vdash\; A}{\Delta \bowtie \Delta_A\,;\,\Omega_L, \Omega_A, A \rightarrowtail B, \Omega_R \;\vdash\; C}\,{\rightarrowtail}L$$

Ordered Uniform Linear Logic Formulas

$$
\begin{array}{rcllcl}
D & ::= & P & \mid & \forall x.D \\
 & \mid & \top & \mid & D \,\&\, D \\
 & \mid & G \twoheadrightarrow D & \mid & G \rightarrowtail D \\
 & \mid & G \multimap D & \mid & G \rightarrow D
\end{array}
$$

$$
\begin{array}{rcllcllcl}
G & ::= & P & \mid & \forall x.G & \mid & \exists x.G \\
 & \mid & \top & \mid & G \,\&\, G & & \\
 & \mid & 0 & \mid & G \oplus G & & \\
 & \mid & 1 & \mid & G \bullet G & \mid & G \circ G \\
 & \mid & D \twoheadrightarrow G & \mid & D \rightarrowtail G & \mid & ¡G \\
 & \mid & D \multimap G & \mid & !G & \mid & D \rightarrow G
\end{array}
$$

Ordered Uniform Linear Logic Derivations

$$\Gamma; \Delta; \Omega \vdash G \qquad\qquad \Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P$$

focussed judgment represents

$$\Gamma; \Delta; \Omega_L, D, \Omega_R \vdash P$$

Ordered Uniform Linear Logic Derivations

$\Gamma; \Delta; \Omega \vdash G \qquad \Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P$

$$\frac{\Gamma; \Delta_0; \Omega_0 \vdash G_0 \qquad \Gamma; \Delta_1; \Omega_1 \vdash G_1}{\Gamma; \Delta_0 \bowtie \Delta_1; \Omega_0, \Omega_1 \vdash G_0 \bullet G_1} \bullet R$$

$$\frac{\Gamma; \Delta_0; \Omega_0 \vdash G_0 \qquad \Gamma; \Delta_1; \Omega_1 \vdash G_1}{\Gamma; \Delta_0 \bowtie \Delta_1; \Omega_1, \Omega_0 \vdash G_0 \circ G_1} \circ R$$

$$\frac{\Gamma; \Delta; \Omega, D \vdash G}{\Gamma; \Delta; \Omega \vdash D \twoheadrightarrow G} \twoheadrightarrow R \qquad \frac{\Gamma; \Delta; D, \Omega \vdash G}{\Gamma; \Delta; \Omega \vdash D \rightarrowtail G} \rightarrowtail R$$

$$\frac{\Gamma; \Delta, D; \Omega \vdash G}{\Gamma; \Delta; \Omega \vdash D \multimap G} \multimap R \qquad \frac{\Gamma, D; \Delta; \Omega \vdash G}{\Gamma; \Delta; \Omega \vdash D \rightarrow G} \rightarrow R$$

Ordered Uniform Linear Logic Derivations

$$\Gamma; \Delta; \Omega \vdash G \qquad\qquad \Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P$$

$$\frac{\Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P}{\Gamma; \Delta; \Omega_L, D, \Omega_R \vdash P} \mathit{choice}_\Omega$$

$$\frac{\Gamma; \Delta_L, \Delta_R; (\Omega_L; \Omega_R) \vdash D \gg P}{\Gamma; \Delta_L \bowtie D, \Delta_R; \Omega_L, \Omega_R \vdash P} \mathit{choice}_\Delta$$

$$\frac{\Gamma \bowtie D; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P}{\Gamma \bowtie D; \Delta; \Omega_L, \Omega_R \vdash P} \mathit{choice}_\Gamma$$

Ordered Uniform Linear Logic Derivations

$$\Gamma; \Delta; \Omega \vdash G \qquad\qquad \Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P$$

$$\frac{\Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P \qquad \Gamma; \Delta_G; \Omega_G \vdash G}{\Gamma; \Delta_G \bowtie \Delta; (\Omega_L; \Omega_G, \Omega_R) \vdash G \twoheadrightarrow D \gg P} \twoheadrightarrow_L$$

$$\frac{\Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P \qquad \Gamma; \Delta_G; \Omega_G \vdash G}{\Gamma; \Delta_G \bowtie \Delta; (\Omega_L, \Omega_G; \Omega_R) \vdash G \rightarrowtail D \gg P} \rightarrowtail_L$$

$$\frac{\Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P \qquad \Gamma; \Delta_G; \cdot \vdash G}{\Gamma; \Delta_G \bowtie \Delta; (\Omega_L; \Omega_R) \vdash G \multimap D \gg P} \multimap_L$$

$$\frac{\Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P \qquad \Gamma; \cdot; \cdot \vdash G}{\Gamma; \Delta; (\Omega_L; \Omega_R) \vdash G \rightarrow D \gg P} \rightarrow_L$$

Outline

Meta-Circular Interpreter: Pure Linear Logic

Pure Linear Logic: $\Delta \vdash G \qquad \Delta \vdash D \gg P$

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Pure Linear Logic: $\Delta \vdash G \qquad \Delta \vdash D \gg P$

$$\frac{\Delta, D \vdash G}{\Delta \vdash D \multimap G} \qquad \frac{\Delta \vdash D \gg P}{\Delta \bowtie D \vdash P}$$

Meta-Circular Interpreter: Pure Linear Logic

Pure Linear Logic: $\Delta \vdash G \qquad \Delta \vdash D \gg P$

$$\frac{\Delta, D \vdash G}{\Delta \vdash D \multimap G} \qquad \frac{\Delta \vdash D \gg P}{\Delta \bowtie D \vdash P}$$

$$\frac{}{\cdot \vdash P \gg P} \qquad \frac{\Delta \vdash D \gg P \quad \Delta_G \vdash G}{\Delta \bowtie \Delta_G \vdash G \multimap D \gg P}$$

Meta-Circular Interpreter: Encoding

```
frm : type.                  atom : type.
atm : atom -> frm.           =o : frm -> frm -> frm.

hyp : frm -> o.              goal : frm -> o.
focus : frm -> atom -> o.
```

$$\Delta \vdash G \qquad \Delta \vdash D \gg P$$

```
goal G.     focus D P.
```

Meta-Circular Interpreter: Encoding

```
frm  :  type.                     atom  :  type.
atm  :  atom -> frm.              =o  :  frm -> frm -> frm.

hyp  :  frm -> o.                 goal  :  frm -> o.
focus  :  frm -> atom -> o.
```

```
goal (D =o G) o- (hyp D -o goal G).
```

$$\frac{\Delta, D \vdash G}{\Delta \vdash D \multimap G}$$

Meta-Circular Interpreter: Encoding

```
frm : type.                    atom : type.
atm : atom -> frm.             =o : frm -> frm -> frm.

hyp : frm -> o.                goal : frm -> o.
focus : frm -> atom -> o.
```

```
goal (D =o G) o- (hyp D -o goal G).

goal (atm P) o- hyp D, focus D P.
```

$$\frac{\Delta \;\vdash\; D \gg P}{\Delta \bowtie D \vdash\; P}$$

Meta-Circular Interpreter: Encoding

```
frm   : type.                  atom  : type.
atm   : atom -> frm.           =o  : frm -> frm -> frm.

hyp   : frm -> o.              goal  : frm -> o.
focus : frm -> atom -> o.
```

```
goal (D =o G) o- (hyp D -o goal G).

goal (atm P) o- hyp D, focus D P.

focus (atm P) P.
```

$$\frac{}{\cdot \vdash P \gg P}$$

Meta-Circular Interpreter: Encoding

```
frm  : type.                atom : type.
atm  : atom -> frm.         =o : frm -> frm -> frm.

hyp  : frm -> o.            goal : frm -> o.
focus : frm -> atom -> o.
```

```
goal (D =o G) o- (hyp D -o goal G).

goal (atm P) o- hyp D, focus D P.

focus (atm P) P.

focus (G =o D) P o- focus D P, goal G.
```

$$\frac{\Delta \vdash D \gg P \quad \Delta_G \vdash G}{\Delta \bowtie \Delta_G \vdash G \multimap D \gg P}$$

Meta-Circular Interpreter: Encoding

```
frm   : type.                    atom  : type.
atm   : atom -> frm.             =o  : frm -> frm -> frm.
=>  : frm -> frm -> frm.         bang  : frm -> frm.

hyp   : frm -> o.                goal  : frm -> o.
focus  : frm -> atom -> o.
```

$$\Gamma; \Delta \vdash G \qquad \Gamma; \Delta \vdash D \gg P$$

```
goal G.     focus D P.
```

Meta-Circular Interpreter: Encoding

```
frm  : type.                  atom : type.
atm  : atom -> frm.           =o   : frm -> frm -> frm.
=>   : frm -> frm -> frm.     bang : frm -> frm.

hyp  : frm -> o.              goal : frm -> o.
focus : frm -> atom -> o.
```

```
goal (D => G) o- (hyp D -> goal G).
```

$$\frac{\Gamma, D; \Delta \vdash G}{\Gamma; \Delta \vdash D \to G}$$

Meta-Circular Interpreter: Encoding

```
frm : type.                      atom : type.
atm : atom -> frm.               =o : frm -> frm -> frm.
=> : frm -> frm -> frm.          bang : frm -> frm.

hyp : frm -> o.                  goal : frm -> o.
focus : frm -> atom -> o.
```

```
goal (D => G) o- (hyp D -> goal G).

focus (G => D) P o- focus D P, goal (bang G).
```

$$\frac{\Gamma;\Delta \vdash D \gg P \quad \Gamma;\cdot \vdash G}{\Gamma;\Delta \vdash G \to D \gg P}$$

Meta-Circular Interpreter: Encoding

```
frm  : type.                  atom : type.
atm  : atom -> frm.           =o : frm -> frm -> frm.
=> : frm -> frm -> frm.       bang : frm -> frm.

hyp  : frm -> o.              goal : frm -> o.
focus : frm -> atom -> o.
```

```
goal (D => G) o- (hyp D -> goal G).

focus (G => D) P o- focus D P, goal (bang G).

goal (bang G) o- !G.
```

$$\frac{\Gamma; \cdot \vdash G}{\Gamma; \cdot \vdash\ !G}$$

Meta-Circular Interpreter: Ordered Linear Logic

Ordered Linear Logic:

$\Gamma;\Delta;\Omega \vdash G \qquad \Gamma;\Delta;(\Omega_L;\Omega_R) \vdash D \gg P$

Meta-Circular Interpreter: Ordered Linear Logic

Ordered Linear Logic:

$\Gamma; \Delta; \Omega \vdash G$ $\quad\quad$ $\Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P$

Problem: No way to represent split ordered context.

Meta-Circular Interpreter: Ordered Linear Logic

Ordered Linear Logic:

$\Gamma; \Delta; \Omega \vdash G \qquad \Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P$

Problem: No way to represent split ordered context.

Solution: Remove need for splitting ordered context.

Outline

Residuation

Logically "compile" clause into new goal.

Removes need to split ordered context when focussing on non-ordered clause.

$$\frac{\Gamma; \Delta_L, \Delta_R; (\Omega_L; \Omega_R) \vdash D \gg P}{\Gamma; \Delta_L, D, \Delta_R; \Omega_L, \Omega_R \vdash P} choice_\Delta$$

$$\frac{\Gamma \bowtie D; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P}{\Gamma \bowtie D; \Delta; \Omega_L, \Omega_R \vdash P} choice_\Gamma$$

Residuation

Logically "compile" clause into new goal.

$$G_I \,;\, D \gg P \setminus G_O$$

Residuation

Logically "compile" clause into new goal.

$$G_I \,;\, D \gg P \setminus G_O$$

$$\frac{}{G \,;\, P \gg P \setminus G} \qquad \frac{G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, \forall x.D \gg P \setminus \exists x.G_O}$$

Residuation

Logically "compile" clause into new goal.

$$G_I \,;\, D \gg P \setminus G_O$$

$$\frac{}{G \,;\, P \gg P \setminus G} \qquad \frac{G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, \forall x.D \gg P \setminus \exists x.G_O}$$

$$\frac{}{G \,;\, \top \gg P \setminus 0} \qquad \frac{G_I \,;\, D_0 \gg P \setminus G_0 \qquad G_I \,;\, D_1 \gg P \setminus G_1}{G_I \,;\, D_0 \mathbin{\&} D_1 \gg P \setminus G_0 \oplus G_1}$$

Residuation

Logically "compile" clause into new goal.

$$G_I \,;\, D \gg P \setminus G_O$$

$$\frac{}{G \,;\, P \gg P \setminus G} \qquad \frac{G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, \forall x.D \gg P \setminus \exists x.G_O}$$

$$\frac{}{G \,;\, \top \gg P \setminus 0} \qquad \frac{G_I \,;\, D_0 \gg P \setminus G_0 \qquad G_I \,;\, D_1 \gg P \setminus G_1}{G_I \,;\, D_0 \,\&\, D_1 \gg P \setminus G_0 \oplus G_1}$$

$$\frac{G \circ G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, G \twoheadrightarrow D \gg P \setminus G_O} \qquad \frac{G \bullet G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, G \rightarrowtail D \gg P \setminus G_O}$$

Residuation

Logically "compile" clause into new goal.

$$G_I \,;\, D \gg P \setminus G_O$$

$$\frac{}{G \,;\, P \gg P \setminus G} \qquad \frac{G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, \forall x.D \gg P \setminus \exists x.G_O}$$

$$\frac{}{G \,;\, \top \gg P \setminus 0} \qquad \frac{G_I \,;\, D_0 \gg P \setminus G_0 \qquad G_I \,;\, D_1 \gg P \setminus G_1}{G_I \,;\, D_0 \,\&\, D_1 \gg P \setminus G_0 \oplus G_1}$$

$$\frac{G \circ G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, G \twoheadrightarrow D \gg P \setminus G_O} \qquad \frac{G \bullet G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, G \rightarrowtail D \gg P \setminus G_O}$$

$$\frac{¡G \bullet G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, G \multimap D \gg P \setminus G_O} \qquad \frac{!G \bullet G_I \,;\, D \gg P \setminus G_O}{G_I \,;\, G \rightarrow D \gg P \setminus G_O}$$

Residuation

Logically "compile" clause into new goal.

$$G_I \,;\, D \gg P \setminus G_O$$

New choice rules:

$$\frac{1 \,;\, D \gg P \setminus G \qquad \Gamma; \Delta; \Omega \vdash G}{\Gamma; \Delta \bowtie D; \Omega \vdash P} \mathit{choice}_\Delta$$

$$\frac{1 \,;\, D \gg P \setminus G \qquad \Gamma \bowtie D; \Delta; \Omega \vdash G}{\Gamma \bowtie D; \Delta; \Omega \vdash P} \mathit{choice}_\Gamma$$

No split ordered contexts.

Remove Ordered Choice

$$\frac{\Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P}{\Gamma; \Delta; \Omega_L, D, \Omega_R \vdash P} choice_\Omega$$

- We cannot residuate away ordered choice context split.

Remove Ordered Choice

$$\frac{\Gamma; \Delta; (\Omega_L; \Omega_R) \vdash D \gg P}{\Gamma; \Delta; \Omega_L, D, \Omega_R \vdash P} \mathit{choice}_\Omega$$

- We cannot residuate away ordered choice context split.
- So let's just remove ordered choice entirely.

Use Placeholders

$$\cdot\,;\ \cdot\,;\ P_2,\ P_1 \twoheadrightarrow P_2 \rightarrowtail P,\ P_1 \vdash P$$

Use Placeholders

$$\cdot\,;\ \cdot\,;\ P_2,\ P_1 \twoheadrightarrow P_2 \rightarrowtail P,\ P_1 \vdash P$$

can be transformed to

$$\cdot\,;\ Q_P \twoheadrightarrow P_1 \twoheadrightarrow P_2 \rightarrowtail P\,;\ P_2,\ Q_P,\ P_1 \vdash P$$

Use Placeholders

$$\frac{\Xi \qquad \cdot;\cdot;\ P_2, Q_P, P_1 \vdash P_2 \bullet P_1 \circ Q_P \circ 1}{\cdot; Q_P \twoheadrightarrow P_1 \twoheadrightarrow P_2 \rightarrowtail P; P_2, Q_P, P_1 \vdash P}\ choice_{\Delta}$$

$$\Xi = 1\ ;\ Q_P \twoheadrightarrow P_1 \twoheadrightarrow P_2 \rightarrowtail P \gg P \setminus P_2 \bullet P_1 \circ Q_P \circ 1$$

Use Placeholders

$$\dfrac{\Xi \quad \dfrac{\cdot;\cdot;P_2 \vdash P_2 \qquad \cdot;\cdot;Q_P, \boxed{P_1} \vdash \boxed{P_1} \circ Q_P \circ 1}{\cdot;\cdot;P_2, Q_P, P_1 \vdash P_2 \bullet P_1 \circ Q_P \circ 1}\bullet_R}{\cdot; Q_P \twoheadrightarrow P_1 \twoheadrightarrow P_2 \rightarrowtail P; P_2, Q_P, P_1 \vdash P}\,\mathit{choice}_\Delta$$

$$\Xi = 1\,;\; Q_P \twoheadrightarrow P_1 \twoheadrightarrow P_2 \rightarrowtail P \gg P \setminus P_2 \bullet P_1 \circ Q_P \circ 1$$

Use Placeholders

$$\dfrac{\Xi \qquad \dfrac{\cdot;\cdot;P_2 \vdash P_2 \qquad \dfrac{\cdot;\cdot;P_1 \vdash P_1 \qquad \cdot;\cdot;Q_P \vdash Q_P \circ 1}{\cdot;\cdot;Q_P, P_1 \vdash P_1 \circ Q_P \circ 1}\circ_R}{\cdot;\cdot;P_2, Q_P, P_1 \vdash P_2 \bullet P_1 \circ Q_P \circ 1}\bullet_R}{\cdot; Q_P \twoheadrightarrow P_1 \twoheadrightarrow P_2 \rightarrowtail P; P_2, Q_P, P_1 \vdash P}\mathit{choice}_\Delta$$

$$\Xi = 1\,;\ Q_P \twoheadrightarrow P_1 \twoheadrightarrow P_2 \rightarrowtail P \gg P \setminus P_2 \bullet P_1 \circ Q_P \circ 1$$

Outline

Uniform Atomic Ordered Linear Logic Syntax

- Distinguished placeholder predicate: Q_X (x is a term).

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- Extend goal formulae with placeholders:

$$G ::= Q_x \mid P \mid \ldots$$

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- New kind of (modified) clause formulae:

$$E ::= D \mid Q_x \twoheadrightarrow D$$

Uniform Atomic Ordered Linear Logic Syntax

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- Demoted ordered context:

$$\omega \quad ::= \quad \cdot \mid \omega, \; Q_x \qquad \text{where } x \text{ not in } \omega$$

Uniform Atomic Ordered Linear Logic Syntax

- Distinguished placeholder predicate: Q_X (x is a term).
- Extend goal formulae with placeholders:

$$G \quad ::= \quad Q_x \mid P \mid \ldots$$

- New kind of (modified) clause formulae:

$$E \quad ::= \quad D \mid Q_x \twoheadrightarrow D$$

- Demoted ordered context:

$$\omega \quad ::= \quad \cdot \mid \omega, Q_x \qquad \text{where } x \text{ not in } \omega$$

- Modified linear context:

$$\delta \quad ::= \quad \cdot \mid \delta, E$$

Uniform Atomic Ordered Linear Logic Derivations

$$\Gamma; \delta; \omega \vdash G$$

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$$\Gamma; \delta; \omega \vdash G$$

$$\frac{\Gamma; \delta, Q_x \twoheadrightarrow D; \omega, Q_x \vdash G}{\Gamma; \delta; \omega \vdash D \twoheadrightarrow G} \twoheadrightarrow_{R'} (x \texttt{ new})$$

Uniform Atomic Ordered Linear Logic Derivations

$$\Gamma; \delta; \omega \vdash G$$

$$\frac{\Gamma; \delta, Q_x \twoheadrightarrow D; \omega, Q_x \vdash G}{\Gamma; \delta; \omega \vdash D \twoheadrightarrow G} \twoheadrightarrow_{R'} \ (x \ \texttt{new})$$

$$\frac{\Gamma; \delta, Q_x \twoheadrightarrow D; Q_x, \omega \vdash G}{\Gamma; \delta; \omega \vdash D \rightarrowtail G} \rightarrowtail_{R'} \ (x \ \texttt{new})$$

Uniform Atomic Ordered Linear Logic Derivations

$$\Gamma; \delta; \omega \vdash G$$

$$\frac{\Gamma; \delta, Q_x \twoheadrightarrow D; \omega, Q_x \vdash G}{\Gamma; \delta; \omega \vdash D \twoheadrightarrow G} \twoheadrightarrow_{R'} \ (x \ \texttt{new})$$

$$\frac{\Gamma; \delta, Q_x \twoheadrightarrow D; Q_x, \omega \vdash G}{\Gamma; \delta; \omega \vdash D \rightarrowtail G} \rightarrowtail_{R'} \ (x \ \texttt{new})$$

$$\frac{}{\Gamma; \cdot; Q_x \vdash Q_x} \mathit{choice}_\omega$$

Uniform Atomic Ordered Linear Logic Derivations

$$\Gamma; \delta; \omega \vdash G$$

$$\frac{\Gamma; \delta, Q_x \twoheadrightarrow D; \omega, Q_x \vdash G}{\Gamma; \delta; \omega \vdash D \twoheadrightarrow G} \twoheadrightarrow_{R'} \ (x \ \texttt{new})$$

$$\frac{\Gamma; \delta, Q_x \twoheadrightarrow D; Q_x, \omega \vdash G}{\Gamma; \delta; \omega \vdash D \rightarrowtail G} \rightarrowtail_{R'} \ (x \ \texttt{new})$$

$$\frac{}{\Gamma; \cdot; Q_x \vdash Q_x} \mathit{choice}_\omega$$

$$\frac{1 \, ; \, E \gg P \setminus G \qquad \Gamma; \delta; \omega \vdash G}{\Gamma; \delta \bowtie E; \omega \vdash P} \mathit{choice}_\delta$$

Uniform Atomic Ordered Linear Logic Derivations

$$\Gamma; \delta; \omega \vdash G$$

$$\frac{\Gamma; \delta, Q_x \twoheadrightarrow D; \omega, Q_x \vdash G}{\Gamma; \delta; \omega \vdash D \twoheadrightarrow G} \twoheadrightarrow_{R'} \ (x \ \mathtt{new})$$

$$\frac{\Gamma; \delta, Q_x \twoheadrightarrow D; Q_x, \omega \vdash G}{\Gamma; \delta; \omega \vdash D \rightarrowtail G} \rightarrowtail_{R'} \ (x \ \mathtt{new})$$

$$\frac{}{\Gamma; \cdot; Q_x \vdash Q_x} \mathit{choice}_\omega$$

$$\frac{1 \,;\, E \gg P \setminus G \qquad \Gamma; \delta; \omega \vdash G}{\Gamma; \delta \bowtie E; \omega \vdash P} \mathit{choice}_\delta$$

$$\frac{1 \,;\, E \gg P \setminus G \qquad \Gamma \bowtie E; \delta; \omega \vdash G}{\Gamma \bowtie E; \delta; \omega \vdash P} \mathit{choice}_\Gamma$$

Outline

Ordered Linear Logic

Meta-Circular Interpreters

Unsplitting Ordered Contexts

Uniform Atomic Ordered Linear Logic

Meta-Circular Interpreter for Olli

OLL syntax in Olli

```
trm : type.
atom : type
place : trm -> atm.
# : frm -> frm -> frm.
& : frm -> frm -> frm.
forall : (trm -> frm) -> frm.
->> : frm -> frm -> frm.
--o : frm -> frm -> frm.
* : frm -> frm -> frm.
gnab : frm -> frm.
```

```
frm : type.
atm : atom -> frm.
one : frm.
zero : frm.
top : frm.
exists : (trm -> frm) -> frm.
>-> : frm -> frm -> frm.
--> : frm -> frm -> frm.
<> : frm -> frm -> frm.
bang : frm -> frm.
```

$\# \equiv \oplus \qquad * \equiv \bullet \qquad \texttt{<>} \equiv \circ$

Encoding of Residuation

resid : frm −> frm −> atm −> frm −> o.

Encoding of Residuation

resid : frm –> frm –> atm –> frm –> o.

resid G (atm P) P G.

Encoding of Residuation

```
resid : frm -> frm -> atm -> frm -> o.

resid G (atm P) P G.

resid G top P zero.
resid G (D0 & D1) P (G0 # G1) ↞
      resid G D0 P G0 • resid G D1 P G1.
```

Encoding of Residuation

resid : frm −> frm −> atm −> frm −> o.

resid G (atm P) P G.

resid G top P zero.
resid G (D0 & D1) P (G0 # G1) $\twoheadleftarrow$
  resid G D0 P G0 $\bullet$ *resid G D1 P G1.*

resid Gi (forall D) P (exists Go) $\twoheadleftarrow$ $\forall y$. *resid Gi (D y) P (Go y).*

Encoding of Residuation

```
resid : frm -> frm -> atm -> frm -> o.

resid G (atm P) P G.

resid G top P zero.
resid G (D0 & D1) P (G0 # G1) ↞
      resid G D0 P G0 • resid G D1 P G1.

resid Gi (forall D) P (exists Go) ↞ ∀y . resid Gi (D y) P (Go y).

resid Gi (G ->> D) P Go ↞ resid (G <> Gi) D P Go.
```

Encoding of Residuation

```
resid : frm -> frm -> atm -> frm -> o.

resid G (atm P) P G.

resid G top P zero.
resid G (D0 & D1) P (G0 # G1) ↞
        resid G D0 P G0 • resid G D1 P G1.

resid Gi (forall D) P (exists Go) ↞ ∀y . resid Gi (D y) P (Go y).

resid Gi (G ->> D) P Go ↞ resid (G <> Gi) D P Go.
resid Gi (G >-> D) P Go ↞ resid (G * Gi) D P Go.
```

Encoding of Residuation

```
resid : frm -> frm -> atm -> frm -> o.

resid G (atm P) P G.

resid G top P zero.
resid G (D0 & D1) P (G0 # G1) ↞
      resid G D0 P G0 • resid G D1 P G1.

resid Gi (forall D) P (exists Go) ↞ ∀y . resid Gi (D y) P (Go y).

resid Gi (G ->> D) P Go ↞ resid (G <> Gi) D P Go.
resid Gi (G >-> D) P Go ↞ resid (G * Gi) D P Go.
resid Gi (G --o D) P Go ↞ resid (gnab G * Gi) D P Go.
```

Encoding of Residuation

```
resid : frm -> frm -> atm -> frm -> o.

resid G (atm P) P G.

resid G top P zero.
resid G (D0 & D1) P (G0 # G1) ↞
      resid G D0 P G0 • resid G D1 P G1.

resid Gi (forall D) P (exists Go) ↞ ∀y . resid Gi (D y) P (Go y).

resid Gi (G ->> D) P Go ↞ resid (G <> Gi) D P Go.
resid Gi (G >-> D) P Go ↞ resid (G * Gi) D P Go.
resid Gi (G --o D) P Go ↞ resid (gnab G * Gi) D P Go.
resid Gi (G --> D) P Go ↞ resid (bang G * Gi) D P Go.
```

Encoding of Derivations

hyp : frm -> o. *goal : frm -> o.*

Encoding of Derivations

hyp : frm −> o.   *goal : frm −> o.*

goal top $\twoheadleftarrow$ $\top$.
goal (G0 & G1) $\twoheadleftarrow$ *goal G0* & *goal G1.*
goal (G0 # G1) $\twoheadleftarrow$ *goal G0* $\oplus$ *goal G1.*

Encoding of Derivations

hyp : frm −> o. *goal : frm −> o.*

$goal\ top \twoheadleftarrow \top.$
$goal\ (G0\ \&\ G1) \twoheadleftarrow goal\ G0\ \&\ goal\ G1.$
$goal\ (G0\ \#\ G1) \twoheadleftarrow goal\ G0 \oplus goal\ G1.$

$goal\ (forall\ G) \twoheadleftarrow \forall x\ .\ goal\ (G\ x).$
$goal\ (exists\ G) \twoheadleftarrow goal\ (G\ X).$

Encoding of Derivations

$hyp : frm \; -> \; o.$ $\qquad$ $goal : frm \; -> \; o.$

$goal\ top \twoheadleftarrow \top.$
$goal\ (G0\ \&\ G1) \twoheadleftarrow goal\ G0\ \&\ goal\ G1.$
$goal\ (G0\ \#\ G1) \twoheadleftarrow goal\ G0 \oplus goal\ G1.$

$goal\ (forall\ G) \twoheadleftarrow \forall x\ .\ goal\ (G\ x).$
$goal\ (exists\ G) \twoheadleftarrow goal\ (G\ X).$

$goal\ (gnab\ G) \twoheadleftarrow\ ¡\ (goal\ G).$
$goal\ (bang\ G) \twoheadleftarrow\ !\ (goal\ G).$

Encoding of Derivations

hyp : frm -> o. *goal : frm -> o.*

goal top $\leftarrow\!\!\!\!\leftarrow$ $\top$.
goal (G0 & G1) $\leftarrow\!\!\!\!\leftarrow$ *goal G0* $\&$ *goal G1.*
goal (G0 # G1) $\leftarrow\!\!\!\!\leftarrow$ *goal G0* $\oplus$ *goal G1.*

goal (forall G) $\leftarrow\!\!\!\!\leftarrow$ $\forall x$. *goal (G x).*
goal (exists G) $\leftarrow\!\!\!\!\leftarrow$ *goal (G X).*

goal (gnab G) $\leftarrow\!\!\!\!\leftarrow$ ¡ *(goal G).*
goal (bang G) $\leftarrow\!\!\!\!\leftarrow$! *(goal G).*

goal one $\leftarrow\!\!\!\!\leftarrow$ 1.
*goal (G * H)* $\leftarrow\!\!\!\!\leftarrow$ *goal G* $\bullet$ *goal H.*
goal (G <> H) $\leftarrow\!\!\!\!\leftarrow$ *goal G* $\circ$ *goal H.*

Encoding of Derivations Continued

$$\textit{goal (D ->> G)} \twoheadleftarrow$$
$$\forall x\,.\;\; \textit{hyp (atm (place x) ->> D)} \multimap \textit{hyp (atm (place x))} \twoheadrightarrow \textit{goal G.}$$

Encoding of Derivations Continued

$$goal\ (D \ {-}{>}{>}\ G) \twoheadleftarrow$$
$$\quad \forall x\,.\ hyp\ (atm\ (place\ x)\ {-}{>}{>}\ D) \multimap hyp\ (atm\ (place\ x)) \twoheadrightarrow goal\ G.$$

$$goal\ (D \ {>}{-}{>}\ G) \twoheadleftarrow$$
$$\quad \forall x\,.\ hyp\ (atm\ (place\ x)\ {-}{>}{>}\ D) \multimap hyp\ (atm\ (place\ x)) \rightarrowtail goal\ G.$$

Encoding of Derivations Continued

$$goal\ (D\ {-}{>}{>}\ G) \twoheadleftarrow$$
$$\forall x\ .\ \ hyp\ (atm\ (place\ x)\ {-}{>}{>}\ D) \multimap hyp\ (atm\ (place\ x)) \twoheadrightarrow goal\ G.$$

$$goal\ (D\ {>}{-}{>}\ G) \twoheadleftarrow$$
$$\forall x\ .\ \ hyp\ (atm\ (place\ x)\ {-}{>}{>}\ D) \multimap hyp\ (atm\ (place\ x)) \rightarrowtail goal\ G.$$

$$goal\ (D\ {-}{-}{\circ}\ G) \twoheadleftarrow\ \ hyp\ D \multimap goal\ G.$$
$$goal\ (D\ {-}{-}{>}\ G) \twoheadleftarrow\ \ hyp\ D \rightarrow goal\ G.$$

Encoding of Derivations Continued

$$goal\ (D \mathbin{-{>}{>}} G) \twoheadleftarrow$$
$$\forall x\,.\ \ hyp\ (atm\ (place\ x) \mathbin{-{>}{>}} D) \multimap hyp\ (atm\ (place\ x)) \twoheadrightarrow goal\ G.$$

$$goal\ (D \mathbin{{>}{-}{>}} G) \twoheadleftarrow$$
$$\forall x\,.\ \ hyp\ (atm\ (place\ x) \mathbin{-{>}{>}} D) \multimap hyp\ (atm\ (place\ x)) \rightarrowtail goal\ G.$$

$$goal\ (D \mathbin{{-}{-}\circ} G) \twoheadleftarrow\ \ hyp\ D \multimap goal\ G.$$
$$goal\ (D \mathbin{{-}{-}{>}} G) \twoheadleftarrow\ \ hyp\ D \rightarrow goal\ G.$$

$$goal\ (atm\ P) \twoheadleftarrow\ \ hyp\ D \bullet\ resid\ one\ D\ P\ G \bullet\ goal\ G.$$